$$y = \beta_1 x_1 + \cdots + \beta_{K-1} x_{K-1} + \beta_K x_K + \varepsilon$$

Linear Proj

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_{K-1} \\ z_1 \\ \vdots \\ z_M \end{bmatrix},$$

$$E[zx'] = K \iff x_K = \delta_1 x_1 + \cdots + \delta_{K-1} x_{K-1} + \theta_1 z_1 + \cdots + \theta_M z_M + r$$

$\theta_1, \cdots, \theta_M$ 至少有一个 $\neq 0$.

Proof

$$x_K = \hat{x}_K + r_K, \quad \hat{x}_K = \delta_1 x_1 + \cdots + \delta_{K-1} x_{K-1} + \theta_1 z_1 + \cdots + \theta_M z_M$$

$$\hat{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{K-1} \\ \hat{x}_K \end{bmatrix}, \quad x = \hat{x} + r$$

$$E[zx'] = E[z(\hat{x}' + r')] = E[z\hat{x}'] \checkmark$$

$$\hat{x} = \underbrace{\boxed{\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ & & \ddots & & \\ 0 & \cdots & \underset{(k-1)}{1} & \cdots & 0 \\ \delta_1 \cdots & \delta_{K-1} & \theta_1 & \cdots & \theta_M \end{bmatrix}}}_{K \times L} \underbrace{z}_{L \times 1} \qquad \equiv \Pi$$

$$E[zx'] = E[zz']\,\Pi'$$

若 $\theta_1, \cdots, \theta_M$ 都 $= 0$，则 Π 的秩 $\leq K-1$