$\hat\beta$ 是 2SLS，$\tilde\beta$ 是其它的 IV 估计量，$\tilde x$ 是 z 的任意线性组合，从 L 维降到 K 维.

工具变量是 z，$L\times 1$

1st OLS 得到的是 $\underset{K\times 1}{x^*} = \underset{K\times L}{\Pi'} z$，$\underset{L\times K}{\Pi} = E[zz']^{-1}E[zx']$

也即 $\Pi' = \begin{bmatrix} \pi_1^{(1)} & \cdots & \pi_1^{(L)} \\ & \cdots & \\ \pi_K^{(1)} & \cdots & \pi_K^{(L)} \end{bmatrix}$，$x_1^* = \pi_1^{(1)} z_1 + \cdots + \pi_1^{(L)} z_L$

$\cdots$

$x_K^* = \pi_K^{(1)} z_1 + \cdots + \pi_K^{(L)} z_L$

$\tilde x = \Gamma' z$，Γ' 是 $K\times L$，$\tilde\beta \to E[\tilde x x']^{-1}E[\tilde x y] = E[\tilde x x']^{-1}E[\tilde x(x'\beta+\varepsilon)]$

$= \beta + E[\tilde x x']^{-1}E[\tilde x \varepsilon]$

$\hat\beta \to E[x^* x']^{-1}E[x^* y] = \beta + E[x^* x']^{-1}E[x^*\varepsilon]$

$= \beta + E[x^* x^{*\prime}]^{-1}E[x^*\varepsilon]$

但 $E[x^* x'] = E[x^*(x^*+r)'] = E[x^* x^{*\prime}]$

$\mathrm{Avar}(\sqrt N(\hat\beta-\beta)) = \sigma^2 E[x^* x^{*\prime}]^{-1}$

$\mathrm{Avar}(\sqrt N(\tilde\beta-\beta)) = \sigma^2 E[\tilde x x']^{-1}E[\tilde x \tilde x']E[x\tilde x']^{-1}$

要证 $\operatorname{Avar}(\sqrt{N}(\tilde{\beta}-\beta)) - \operatorname{Avar}(\sqrt{N}(\hat{\beta}-\beta))$ p.s.d. 半正定.

(positive semidefinite)

$A^{-1} - B^{-1}$ p.s.d. $\iff$ $B - A$ p.s.d.

也即 $E[x^* x^{*\prime}] - E[x \tilde{x}'] \, E[\tilde{x}\tilde{x}']^{-1} E[\tilde{x} x']$ p.s.d.

因为 $E[x\tilde{x}'] = E[(x^*+r)\tilde{x}'] = E[x^* \tilde{x}' + \underset{K\times 1}{r}\, \underset{1\times L}{z'}\, \underset{L\times K}{\Gamma}]$, 而 $E[z' r] = 0$

$= E[x^* \tilde{x}']$

$$E[x^* x^{*\prime}] - E[x^* \tilde{x}'] \, E[\tilde{x}\tilde{x}']^{-1} E[\tilde{x} x^{*\prime}] \quad ①$$

而 $E[L(x^*|\tilde{x})\, L(x^*|\tilde{x})'] = E\Big[\big(E[\tilde{x}\tilde{x}']^{-1} E[\tilde{x} x^{*\prime}]\big)'\tilde{x}\, \big(\tilde{x}' E[\tilde{x}\tilde{x}']^{-1} E[\tilde{x} x^{*\prime}]\big)\Big]$

$= E[x^* \tilde{x}'] \, E[\tilde{x}\tilde{x}']^{-1} E[\cancel{\tilde{x}\tilde{x}'}] \, E[\cancel{\tilde{x}\tilde{x}'}]^{-1} E[\tilde{x} x^{*\prime}]$

因此, ① $= E[x^* x^{*\prime}] - E[L(x^*|\tilde{x})\, L(x^*|\tilde{x})'] = E[e^* e^{*\prime}]$, p.s.d.