Robust Error

$$y_i = x_{i1}\beta_1 + \cdots + x_{iK}\beta_K + \varepsilon_i = \vec{x}_i{}'\beta + \varepsilon_i, \quad \vec{x}_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{iK} \end{bmatrix}$$

$i=1,\cdots,N$ 观测值, $k=1,\cdots,K$ 个自变量.

$$y = X\beta, \quad X = \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & & \\ x_{N1} & \cdots & x_{NK} \end{bmatrix} \overset{K \rightarrow}{} \downarrow N = \begin{bmatrix} \vec{x}_1{}' \\ \vdots \\ \vec{x}_N{}' \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

OLS

$$\min_{\tilde{\beta}} (y - X\tilde{\beta})'(y - X\tilde{\beta}) \Rightarrow \hat{\beta} = (X'X)^{-1}X'y$$

$$X'X = [\vec{x}_1 \cdots \vec{x}_N]\begin{bmatrix} \vec{x}_1{}' \\ \vdots \\ \vec{x}_N{}' \end{bmatrix} = \sum_{i=1}^N \vec{x}_i \vec{x}_i{}'.$$

$\frac{1}{n}\Sigma \rightarrow E$

$$\hat{\beta} = \left(\sum_{i=1}^N \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\sum_{i=1}^N \vec{x}_i y_i\right) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N}\vec{x}_i y_i\right)$$

$$= \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N}\vec{x}_i(\vec{x}_i{}'\beta + \varepsilon_i)\right)$$

$N \uparrow \quad \hat{u}_1$

$$= \beta + \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \varepsilon_i\right)$$

$X'\Omega X \quad i, j$

$$\hat{\sigma}^2 = \frac{1}{n-k}\sum_{i=1}^n \hat{u}_i^2 \qquad \Omega = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$$

$$\xrightarrow[N\to\infty]{} \beta + E(\vec{x}_i\vec{x}_i{}')^{-1}E(x_i\varepsilon_i)$$

$$\equiv \sqrt{N}\frac{1}{N}\sum_{i=1}^N g_i = \sqrt{N}\bar{g}$$

$\sqrt{N}$ 统计量: $\sqrt{N}(\hat{\beta} - \beta) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i{}'\right)^{-1}\underbrace{\left(\sqrt{N}\frac{1}{N}\sum_{i=1}^N \vec{x}_i\varepsilon_i\right)}$ $\quad \hat{u}_1^2$

$\sqrt{N}\bar{g} \xrightarrow{d} N(0, E(\varepsilon_i^2\vec{x}_i\vec{x}_i{}'))$, Central Limit Theorem for Martingale Difference Sequence

$$\sqrt{N}(\hat{\beta} - \beta) \xrightarrow{d} N(0, E(\vec{x}_i\vec{x}_i{}')^{-1}\underline{E(\varepsilon_i^2\vec{x}_i\vec{x}_i{}')}E(\vec{x}_i\vec{x}_i{}')^{-1})$$ Robust variance.

$$\widehat{Avar}(\hat{\beta}) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i{}'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i{}'\underbrace{(y_i - \vec{x}_i{}'\hat{\beta})^2}_{u_i}\right)\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i{}'\right)^{-1}$$

同方差:

$$E[\varepsilon_i^2|\vec{x}_i]=\sigma^2.$$

$$E[\varepsilon_i^2\vec{x}_i\vec{x}_i']=E\left[E[\varepsilon_i^2\vec{x}_i\vec{x}_i'|\vec{x}_i]\right]=E\left[E[\varepsilon_i^2|x_i]\vec{x}_i\vec{x}_i'\right]=\sigma^2E[\vec{x}_i\vec{x}_i']$$

$$\hat{\sigma}^2=\frac{1}{N-K}(y-X\hat{\beta})'(y-X\hat{\beta})$$

$\mathrm{cov}(g_i,g_j)$

$\mathrm{cov}(x_i,x_j)$ 标量

$=\mathrm{cov}(x_j,x_i)$

①式假设了无自相关, $E[\varepsilon_i^2\vec{x}_i\vec{x}_i']$, 不同的 i,j 之间没有关系. K个变量。

Newey-West Estimator

$E[g_ig_i']$

$$x_i=\begin{bmatrix}x_{i1}\\ \vdots\\ x_{iK}\end{bmatrix}\quad K\times1$$

$\mathrm{Var}(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}g_i)$ 若有自相关, 则 $E[g_ig_{i-j}]\neq0$, $g_i=\vec{x}_i\varepsilon_i$.

$\mathrm{cov}(g_i,g_{i+j})\equiv\Gamma_j$, $\mathrm{cov}(g_{i+j},g_i)=\Gamma_{-j}$, $\mathrm{Var}(g_i)=\Gamma_0$

$\Gamma_j=\Gamma_{-j}'$, 因为 g_i 是向量: $\mathrm{cov}(g_i,g_{i+j})=\mathrm{cov}\left(\begin{bmatrix}g_i^1\\ \vdots\\ g_i^K\end{bmatrix}\begin{bmatrix}g_{i+j}^1\\ \vdots\\ g_{i+j}^K\end{bmatrix}\right)$

$$=\begin{bmatrix}\mathrm{cov}(g_i^1,g_{i+j}^1), & \mathrm{cov}(g_i^1,g_{i+j}^2) & \cdots & \mathrm{cov}(g_i^1,g_{i+j}^K)\\ \mathrm{cov}(g_i^2,g_{i+j}^1), & \mathrm{cov}(g_i^2,g_{i+j}^2) & \cdots & \mathrm{cov}(g_i^2,g_{i+j}^K)\\ & \cdots & & \\ \mathrm{cov}(g_i^K,g_{i+j}^1), & \mathrm{cov}(g_i^K,g_{i+j}^2) & \cdots & \mathrm{cov}(g_i^K,g_{i+j}^K)\end{bmatrix}$$

$\mathrm{cov}(g_{i+j},g_i)=$

$$\begin{bmatrix}\mathrm{cov}(g_{i+j}^1,g_i^1), & \mathrm{cov}(g_{i+j}^1,g_i^2) & \cdots & \mathrm{cov}(g_{i+j}^1,g_i^K)\\ \mathrm{cov}(g_{i+j}^2,g_i^1), & \mathrm{cov}(g_{i+j}^2,g_i^2) & \cdots & \mathrm{cov}(g_{i+j}^2,g_i^K)\\ & \cdots & & \\ \mathrm{cov}(g_{i+j}^K,g_i^1), & \mathrm{cov}(g_{i+j}^K,g_i^2) & \cdots & \mathrm{cov}(g_{i+j}^K,g_i^K)\end{bmatrix}$$

$$Var(\sqrt{N}\bar{g}) = Var\left(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N} g_i\right) = \frac{1}{N} Var\left(\sum_{i=1}^{N} g_i\right) = \frac{1}{N} Var(g_1 + \cdots + g_N)$$

$$= \frac{1}{N}\big(N\Gamma_0 + (N-1)\Gamma_1 + (N-2)\Gamma_2 + \cdots + \Gamma_{N-1} + (N-1)\Gamma_{-1} + (N-2)\Gamma_{-2} + \cdots + \Gamma_{-(N-1)}\big)$$

$cov(g_1, g_2)$ $cov(g_1, g_3)$

$cov(g_2, g_3)$ $cov(g_2, g_4)$

$$\lim_{N\to\infty} Var(\sqrt{N}\bar{g}) = \sum_{j=-\infty}^{\infty} \Gamma_j = \Gamma_0 + \sum_{j=1}^{\infty}(\Gamma_j + \Gamma_j')$$

g_1, g_N

Gordin

Gordin's CLT 要求在足够远处自相关消失，因此有截断位置 p.

当 $j > p$ 时，$\Gamma_j = 0$.

(g_2, g_1), (g_3, g_2)

有估计量 $\widehat{Avar}(\sqrt{N}\bar{g}) = \boxed{\hat{\Gamma}_0 + \sum_{j=1}^{p}(\hat{\Gamma}_j + \hat{\Gamma}_j')}$ (Hansen & Singleton (1982), West (1986))

在有限样本下不一定半正定. 调整方法：加权重.

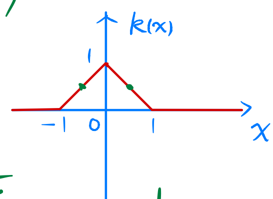
$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} \boxed{k\left(\frac{j}{p(N)}\right)} \hat{\Gamma}_j$$

若 $k(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$，则还原.

kernel function: 非参计量

Newey-West:

$$k(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$, Bartlett type kernel.



例: $p(N) = 3$.

$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} k\left(\frac{j}{3}\right)\hat{\Gamma}_j = \overset{\frac{1}{3}}{k\left(-\frac{2}{3}\right)}\hat{\Gamma}_{-2} + \overset{\frac{2}{3}}{k\left(-\frac{1}{3}\right)}\hat{\Gamma}_{-1} + \overset{1}{k(0)}\hat{\Gamma}_0 + \overset{\frac{2}{3}}{k\left(\frac{1}{3}\right)}\hat{\Gamma}_1 + \overset{\frac{1}{3}}{k\left(\frac{2}{3}\right)}\hat{\Gamma}_2$$

$$= 1 \cdot \hat{\Gamma}_0 + \frac{2}{3}(\hat{\Gamma}_1 + \hat{\Gamma}_1') + \frac{1}{3}(\hat{\Gamma}_2 + \hat{\Gamma}_2')$$

$x_i \varepsilon_i$

$x_j \varepsilon_j$

分量形式：

$$\frac{1}{N}\sum_{i=1}^{N} \hat{\varepsilon}_i^2 \vec{x}_i \vec{x}_i' + \frac{1}{N}\sum_{l=1}^{p}\sum_{i=l+1}^{N} \boxed{k(l)}\,\boxed{\hat{\varepsilon}_i \hat{\varepsilon}_{i-l}}\,(\vec{x}_i \vec{x}_{i-l}' + \vec{x}_{i-l}\vec{x}_i')$$

Newey-West.

NW