Robust Error

$$y_i = x_{i1}\beta_1 + \cdots + x_{iK}\beta_K + \varepsilon_i = \vec{x}_i{}'\beta + \varepsilon_i \,, \quad \vec{x}_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{iK} \end{bmatrix}$$

$i=1,\dots,N$ 观测值，$k=1,\dots,K$ 个自变量.

$$y = X\beta \,, \quad X = \underset{\xrightarrow{\;K\;}}{\begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & & \\ x_{N1} & \cdots & x_{NK} \end{bmatrix}} \downarrow N = \begin{bmatrix} \vec{x}_1{}' \\ \vdots \\ \vec{x}_N{}' \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

OLS

$$\min_{\tilde\beta} (y - X\tilde\beta)'(y - X\tilde\beta) \Rightarrow \hat\beta = (X'X)^{-1}X'y$$

$$X'X = [\vec{x}_1 \cdots \vec{x}_N] \begin{bmatrix} \vec{x}_1{}' \\ \vdots \\ \vec{x}_N{}' \end{bmatrix} = \sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}' .$$

$$\hat\beta = \left(\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\sum_{i=1}^{N} \vec{x}_i y_i\right) = \left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N} \vec{x}_i y_i\right)$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N} \vec{x}_i (\vec{x}_i{}'\beta + \varepsilon_i)\right)$$

$$= \beta + \left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \varepsilon_i\right)$$

$$\xrightarrow[N\to\infty]{} \beta + E(\vec{x}_i \vec{x}_i{}')^{-1} E(x_i \varepsilon_i)$$

$\sqrt{N}$ 统计量: $\sqrt{N}(\hat\beta - \beta) = \left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1} \underbrace{\left(\sqrt{N}\,\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \varepsilon_i\right)}$ $\equiv \sqrt{N}\,\frac{1}{N}\sum_{i=1}^{N} g_i = \sqrt{N}\,\bar{g}$

$\sqrt{N}\,\bar{g} \xrightarrow{d} N(0, E(\varepsilon_i^2 \vec{x}_i \vec{x}_i{}'))$, Central Limit Theorem for Martingale Difference Sequence

$\sqrt{N}(\hat\beta - \beta) \xrightarrow{d} N(0, E(\vec{x}_i \vec{x}_i{}')^{-1} E(\varepsilon_i^2 \vec{x}_i \vec{x}_i{}') E(\vec{x}_i \vec{x}_i{}')^{-1})$ Robust variance.

$$\widehat{Avar}(\hat\beta) = \left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}' (y_i - \vec{x}_i{}'\hat\beta)^2\right)\left(\frac{1}{N}\sum_{i=1}^{N} \vec{x}_i \vec{x}_i{}'\right)^{-1}$$

同方差:

$$E[\varepsilon_i^2 | \vec{x}_i] = \sigma^2 .$$

$$E[\varepsilon_i^2 \vec{x}_i \vec{x}_i'] = E\left[E[\varepsilon_i^2 \vec{x}_i \vec{x}_i' | \vec{x}_i]\right] = E\left[E[\varepsilon_i^2 | x_i]\, \vec{x}_i \vec{x}_i'\right] = \sigma^2 E[\vec{x}_i \vec{x}_i']$$

$$\hat{\sigma}^2 = \frac{1}{N-K}(y - X\hat{\beta})'(y - X\hat{\beta})$$

① 式假设了无自相关, $E[\varepsilon_i^2 \vec{x}_i \vec{x}_i']$, 不同的 i, j 之间没有联系.

Newey - West Estimator

$Var(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N} g_i)$ 若有自相关, 则 $E[g_i g_{i-j}] \neq 0$, $g_i = x_i \varepsilon_i$.

$$cov(g_i, g_{i+j}) \equiv \Gamma_j, \quad cov(g_{i+j}, g_i) = \Gamma_{-j}, \quad Var(g_i) = \Gamma_0$$

$\Gamma_j = \Gamma_{-j}'$, 因为 g_i 是向量: $cov(g_i, g_{i+j}) = cov\left(\begin{bmatrix} g_i^1 \\ \vdots \\ g_i^k \end{bmatrix}, \begin{bmatrix} g_{i+j}^1 \\ \vdots \\ g_{i+j}^k \end{bmatrix}\right)$

$$= \begin{bmatrix} cov(g_i^1, g_{i+j}^1), & cov(g_i^1, g_{i+j}^2) & \cdots & cov(g_i^1, g_{i+j}^k) \\ cov(g_i^2, g_{i+j}^1), & cov(g_i^2, g_{i+j}^2) & \cdots & cov(g_i^2, g_{i+j}^k) \\ & \cdots & & \\ cov(g_i^k, g_{i+j}^1), & cov(g_i^k, g_{i+j}^2) & \cdots & cov(g_i^k, g_{i+j}^k) \end{bmatrix}$$

$cov(g_{i+j}, g_i) =$

$$\begin{bmatrix} cov(g_{i+j}^1, g_i^1), & cov(g_{i+j}^1, g_i^2) & \cdots & cov(g_{i+j}^1, g_i^k) \\ cov(g_{i+j}^2, g_i^1), & cov(g_{i+j}^2, g_i^2) & \cdots & cov(g_{i+j}^2, g_i^k) \\ & \cdots & & \\ cov(g_{i+j}^k, g_i^1), & cov(g_{i+j}^k, g_i^2) & \cdots & cov(g_{i+j}^k, g_i^k) \end{bmatrix}$$

$$Var(\sqrt{N}\bar{g}) = Var(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N} g_i) = \frac{1}{N} Var(\sum_{i=1}^{N} g_i) = \frac{1}{N} Var(g_1 + \cdots + g_N)$$

$$= \frac{1}{N}\Big(N\Gamma_0 + (N-1)\Gamma_1 + (N-2)\Gamma_2 + \cdots + \Gamma_{N-1}$$
$$+ (N-1)\Gamma_{-1} + (N-2)\Gamma_{-2} + \cdots + \Gamma_{-(N-1)} \Big)$$

$$\lim_{N\to\infty} Var(\sqrt{N}\bar{g}) = \sum_{j=-\infty}^{\infty} \Gamma_j = \Gamma_0 + \sum_{j=1}^{\infty}(\Gamma_j + \Gamma_j')$$

Godin's CLT 要求在足够远处自相关消失，因此有截断位置 p，

当 $|j| > p$ 时，$\Gamma_j = 0$.

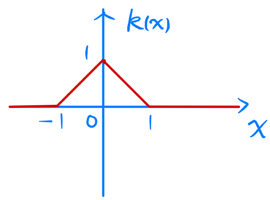
有估计量 $$\widehat{Avar}(\sqrt{N}\bar{g}) = \hat{\Gamma}_0 + \sum_{j=1}^{p}(\hat{\Gamma}_j + \hat{\Gamma}_j')$$ (Hansen & Singleton (1982), West (1986))

在有限样本下不一定半正定. 调整方法：加权重.

$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} k\left(\frac{j}{p(N)}\right)\hat{\Gamma}_j$$ 若 $k(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$，则还原.

Newey – West：

$$k(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$, Bartlett type kernel.



例: $p(N) = 3$.

$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} k\left(\frac{j}{3}\right)\hat{\Gamma}_j = k(-\tfrac{2}{3})\hat{\Gamma}_{-2} + k(-\tfrac{1}{3})\hat{\Gamma}_{-1} + k(0)\hat{\Gamma}_0$$
$$+ k(\tfrac{1}{3})\hat{\Gamma}_1 + k(\tfrac{2}{3})\hat{\Gamma}_2$$
$$= \hat{\Gamma}_0 + \frac{2}{3}(\hat{\Gamma}_1 + \hat{\Gamma}_1') + \frac{1}{3}(\hat{\Gamma}_2 + \hat{\Gamma}_2')$$

分量形式：

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\varepsilon}_i^2 \vec{x}_i \vec{x}_i' + \frac{1}{N}\sum_{l=1}^{p}\sum_{i=l+1}^{N} k(l)\,\hat{\varepsilon}_i\hat{\varepsilon}_{i-l}(\vec{x}_i\vec{x}_{i-l}' + \vec{x}_{i-l}\vec{x}_i')$$