Robust Error

$$y_i = x_{i1}\beta_1 + \cdots + x_{iK}\beta_K + \varepsilon_i = \vec{x}_i'\beta + \varepsilon_i, \quad \vec{x}_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{iK} \end{bmatrix}$$

$i=1,\cdots,N$ 观测值，$k=1,\cdots,K$ 个自变量.

$$y = X\beta, \quad X = \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & & \\ x_{N1} & \cdots & x_{NK} \end{bmatrix} \Big\downarrow N = \begin{bmatrix} \vec{x}_1' \\ \vdots \\ \vec{x}_N' \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

$\xrightarrow{K}$

OLS

$$\min_{\tilde{\beta}} (y - X\tilde{\beta})'(y - X\tilde{\beta}) \Rightarrow \hat{\beta} = (X'X)^{-1}X'y$$

$$X'X = [\vec{x}_1 \cdots \vec{x}_N] \begin{bmatrix} \vec{x}_1' \\ \vdots \\ \vec{x}_N' \end{bmatrix} = \sum_{i=1}^N \vec{x}_i \vec{x}_i'.$$

$$\hat{\beta} = \left(\sum_{i=1}^N \vec{x}_i \vec{x}_i'\right)^{-1}\left(\sum_{i=1}^N \vec{x}_i y_i\right) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i'\right)^{-1}\left(\frac{1}{N}\vec{x}_i y_i\right)$$

$$= \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i'\right)^{-1}\left(\frac{1}{N}\vec{x}_i(\vec{x}_i'\beta + \varepsilon_i)\right)$$

$$= \beta + \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \vec{x}_i'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i \varepsilon_i\right)$$

$$\xrightarrow[N\to\infty]{} \beta + E(\vec{x}_i\vec{x}_i')^{-1}E(x_i\varepsilon_i)$$

$\sqrt{N}$ 统计量: $\sqrt{N}(\hat{\beta}-\beta) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i'\right)^{-1}\underbrace{\left(\sqrt{N}\frac{1}{N}\sum_{i=1}^N \vec{x}_i\varepsilon_i\right)}_{\equiv \sqrt{N}\frac{1}{N}\sum_{i=1}^N g_i = \sqrt{N}\bar{g}}$

$\sqrt{N}\bar{g} \xrightarrow{d} N(0, E(\varepsilon_i^2\vec{x}_i\vec{x}_i'))$, Central Limit Theorem for Martingale Difference Sequence

$\sqrt{N}(\hat{\beta}-\beta) \xrightarrow{d} N(0, E(\vec{x}_i\vec{x}_i')^{-1}E(\varepsilon_i^2\vec{x}_i\vec{x}_i')E(\vec{x}_i\vec{x}_i')^{-1})$ Robust variance.

$$\widehat{Avar}(\hat{\beta}) = \left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i'(y_i - \vec{x}_i'\hat{\beta})^2\right)\left(\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i'\right)^{-1}$$

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同方差:

$$E[\varepsilon_i^2|\vec{x}_i]=\sigma^2.$$

$$E[\varepsilon_i^2\vec{x}_i\vec{x}_i']=E\left[E[\varepsilon_i^2\vec{x}_i\vec{x}_i'|\vec{x}_i]\right]=E\left[E[\varepsilon_i^2|x_i]\vec{x}_i\vec{x}_i'\right]=\sigma^2E[\vec{x}_i\vec{x}_i']$$

$$\hat{\sigma}^2=\frac{1}{N-K}(y-X\hat{\beta})'(y-X\hat{\beta})$$

① 式假设了无自相关，$E[\varepsilon_i^2\vec{x}_i\vec{x}_i']$，不同的 i,j 之间没有关系.

Newey - West Estimator

$Var(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}g_i)$ 若有自相关，则 $E[g_ig_{i-j}]\neq 0$，$g_i=x_i\varepsilon_i$.

$$cov(g_i,g_{i+j})\equiv\Gamma_j,\quad cov(g_{i+j},g_i)=\Gamma_{-j},\quad Var(g_i)=\Gamma_0$$

$\Gamma_j=\Gamma_{-j}'$，因为 g_i 是向量：$cov(g_i,g_{i+j})=cov\left(\begin{bmatrix}g_i^1\\ \vdots\\ g_i^k\end{bmatrix},\begin{bmatrix}g_{i+j}^1\\ \vdots \\ g_{i+j}^k\end{bmatrix}\right)$

$$=\begin{bmatrix}cov(g_i^1,g_{i+j}^1), & cov(g_i^1,g_{i+j}^2) & \cdots & cov(g_i^1,g_{i+j}^k)\\ cov(g_i^2,g_{i+j}^1), & cov(g_i^2,g_{i+j}^2) & \cdots & cov(g_i^2,g_{i+j}^k)\\ & \cdots & & \\ cov(g_i^k,g_{i+j}^1), & cov(g_i^k,g_{i+j}^2) & \cdots & cov(g_i^k,g_{i+j}^k)\end{bmatrix}$$

$cov(g_{i+j},g_i)=$

$$\begin{bmatrix}cov(g_{i+j}^1,g_i^1), & cov(g_{i+j}^1,g_i^2) & \cdots & cov(g_{i+j}^1,g_i^k)\\ cov(g_{i+j}^2,g_i^1), & cov(g_{i+j}^2,g_i^2) & \cdots & cov(g_{i+j}^2,g_i^k)\\ & \cdots & & \\ cov(g_{i+j}^k,g_i^1), & cov(g_{i+j}^k,g_i^2) & \cdots & cov(g_{i+j}^k,g_i^k)\end{bmatrix}$$

$$Var(\sqrt{N}\bar{g}) = Var(\sqrt{N}\frac{1}{N}\sum_{i=1}^{N} g_i) = \frac{1}{N}Var(\sum_{i=1}^{N} g_i) = \frac{1}{N}Var(g_1 + \cdots + g_N)$$

$$= \frac{1}{N}\Big(N\Gamma_0 + (N-1)\Gamma_1 + (N-2)\Gamma_2 + \cdots + \Gamma_{N-1}$$
$$+ (N-1)\Gamma_{-1} + (N-2)\Gamma_{-2} + \cdots + \Gamma_{-(N-1)}\Big)$$

$$\lim_{N\to\infty} Var(\sqrt{N}\bar{g}) = \sum_{j=-\infty}^{\infty}\Gamma_j = \Gamma_0 + \sum_{j=1}^{\infty}(\Gamma_j + \Gamma_j')$$

Godin's CLT 要求在足够远处自相关消失，因此有截断位置 p，

当 $|j|>p$ 时，$\Gamma_j = 0$.

有估计量 $\widehat{Avar}(\sqrt{N}\bar{g}) = \hat{\Gamma}_0 + \sum_{j=1}^{p}(\hat{\Gamma}_j + \hat{\Gamma}_j')$　(Hansen & Singleton (1982), West (1986))

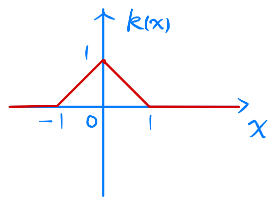
在有限样本下不一定半正定. 调整方法：加权重.

$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} k\left(\frac{j}{p(N)}\right)\hat{\Gamma}_j$$

若 $k(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$，则还原.

Newey–West：

$$k(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$，Bartlett type kernel.



例：$p(N)=3$.

$$\widehat{Avar}(\sqrt{N}\bar{g}) = \sum_{j=-(N-1)}^{N-1} k\left(\frac{j}{3}\right)\hat{\Gamma}_j = k\left(-\frac{2}{3}\right)\hat{\Gamma}_{-2} + k\left(-\frac{1}{3}\right)\hat{\Gamma}_{-1} + k(0)\hat{\Gamma}_0$$
$$+ k\left(\frac{1}{3}\right)\hat{\Gamma}_1 + k\left(\frac{2}{3}\right)\hat{\Gamma}_2$$
$$= \hat{\Gamma}_0 + \frac{2}{3}(\hat{\Gamma}_1 + \hat{\Gamma}_1') + \frac{1}{3}(\hat{\Gamma}_2 + \hat{\Gamma}_2')$$

分量形式：

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\varepsilon}_i^2 \vec{x}_i \vec{x}_i' + \frac{1}{N}\sum_{l=1}^{p}\sum_{i=l+1}^{N} k(l)\,\hat{\varepsilon}_i\hat{\varepsilon}_{i-l}(\vec{x}_i\vec{x}_{i-l}' + \vec{x}_{i-l}\vec{x}_i')$$